Similar matrices

Def: If A and B are $n \times n$ matrices, they are similar if $B = P^{-1}AP$ for some invertible $n \times n$ matrix P. We write $A \sim B$ to mean A is similar to B.

- If $B = P^{-1}AP$, then $PBP^{-1} = A$. If $Q = P^{-1}$, then $A = Q^{-1}BQ$, so $A \sim B \iff B \sim A$.

- Since $A = I^{-1}AI$, we always have $A \sim A$.

- If $A \sim B$ and $B \sim C$, then

$$P^{-1}AP = B \quad \text{and} \quad Q^{-1}BQ = C$$

$$\Rightarrow \quad Q^{-1}(P^{-1}AP)Q = C$$

$$\Rightarrow \quad (PQ)^{-1}A(PQ) = C$$

so $A \sim C$.

Example: A is diagonalizable if and only if $A \sim D$ for a diagonal matrix D.

Thus, if $A \sim B$ and A is diagonalizable, the properties above imply $B \sim D$, so B is diagonalizable as well.

Theorem: If $A \sim B$, then

1.) $\det(A) = \det(B)$ $\left(P^{-1}AP = B \Rightarrow \det P^{-1} \det A \det P = \det B \right)$

2.) $\text{rank}(A) = \text{rank}(B)$

3.) $c_A(x) = c_B(x)$

4.) A and B have same eigenvalues.

5.) $tr(A) = tr(B)$ ($tr(A)$ = trace of A = sum of diagonal)

Ex: If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

$tr(A) = 2 + 2 = 4 = 1 + 3 = tr(B)$,

but $\det(A) = 4 - 1 = 3 \neq 2 = \det(B)$, so A and B are not similar.

Diagonalization

Using new tools from the previous section, we can now say more about eigenvectors and diagonalization

Recall:

If A is an $n \times n$ matrix,

- $c_A(x) = \det(xI - A)$, and eigenvalues are the roots of $c_A(x)$.

- If λ is an eigenvalue of A, then λ-eigenvectors are the nontrivial solutions of

$$(\lambda I - A)\vec{x} = \vec{0}$$

- A is diagonalizable if and only if it has n eigenvectors $\vec{x}_1, \dots, \vec{x}_n$ such that the matrix $P = [\vec{x}_1 \ \cdots \ \vec{x}_n]$ is invertible.

Now we know that P is invertible $\Leftrightarrow$ columns form a basis for $\mathbb{R}^n$, which leads to the following:

Theorem: • A is diagonalizable if and only if $\mathbb{R}^n$ has a basis $\{\vec{x}_1, \dots, \vec{x}_n\}$ consisting of eigenvectors of A.

- In this case, $P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, where each λ_i is the eigenvalue corresponding to $\vec{x}_i$.

Recall that if λ is an eigenvalue of A, an $n \times n$ matrix, the eigenspace of A corresponding to λ is

$$E_\lambda(A) = \{\vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}$$

Equivalently, $E_\lambda(A) = \operatorname{null}(\lambda I - A)$.

This is a subspace of $\mathbb{R}^n$ and consists of all

λ-eigenvectors (plus the 0 vector).

Since $E_\lambda(A) = \text{null}(\lambda I - A)$, we have:

1.) A basis for $E_\lambda(A)$ consists of basic λ-eigenvectors.

2.) $\dim E_\lambda(A) = \#$ of basic λ-eigenvectors $\leq$ multiplicity of λ.

3.) A is diagonalizable if and only if

$\dim(E_\lambda(A)) =$ multiplicity of λ for every eigenvalue λ of A. (Assuming $c_A(x)$ completely factors)

Ex:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$$

$$c_A(x) = \det \begin{bmatrix} x-2 & -1 & -1 \\ -2 & x-1 & 2 \\ 1 & 0 & x+2 \end{bmatrix}$$

$$= 1\big(-2 + (x-1)\big) - 0 + (x+2)\big((x-2)(x-1) - 2\big)$$

$$= (x-3) + (x+2)(x^2 - 3x)$$

$$= (x-3)\big(1 + (x+2)x\big)$$

$$= (x-3)(x^2 + 2x + 1) = (x-3)(x+1)^2$$

Eigenvalues are $\lambda_1 = 3$ (mult 1)

$\lambda_2 = -1$ (mult 2)

$\underline{\lambda_1 = 3}$

$E_3(A) = \text{null}(3I - A)$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & 2 & 2 & 0 \\ 1 & 0 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 \end{array}\right]$$

$\Rightarrow \quad x = -5z$

$\quad\quad y = -6z$

Set $z = t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -5 \\ -6 \\ 1 \end{bmatrix} \quad \text{So} \quad E_3(A) = \text{span}\left\{\begin{bmatrix} -5 \\ -6 \\ 1 \end{bmatrix}\right\}$$

and has dimension 1.

$\underline{\lambda_2 = -1}$:

$$\left[\begin{array}{ccc|c} -3 & -1 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = -z$

$y = 2z$

Set $z = t \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

So $E_{-1}(A) = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$ and has dimension 1.

Thus, its dimension is less than the multiplicity so A is not diagonalizable.

(Also, note that there are at most 2 lin. indep. eigenvectors of A, so not enough to form a basis of $\mathbb{R}^3$.)

Practice problems: 5.5 : 1def, 4